Transitioning from Computational to Abstract Mathematics

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You have just finished calculus and now you're in your first proof-based class. All of the sudden, math is really different. It used to be more or less predictable—if you knew how to read the problem and chose the solution process, you would be okay. With enough practice, you could memorize and understand the process of calculations and do just fine in your classes. "Theorems" used to be these things you would just take for granted as true and you would only ever have to "verify" them with a specific example. Now, you have to not only understand *why* they are true, but you have to prove it. Your math homework looks more like an essay than a problem set.

You might be asking, "Why are there more words on my page than numbers?" It seems weird right now, but once you learn proofs, your power to solve problems will increase ten-fold. Right now, you can solve individual problems, but once you learn to prove, you can capture patterns and find ways to solve all sorts of problems in that category. This handout is intended to be a supplement to your first proof based class. This is a short summary and categorization, not an in-depth resource. For additional resources, see the end of the handout or your professor.

<u>Basics</u>:

- *Definition*: A definition is a description of a mathematical term.
    - <u>Example</u>: A "rational number" is a number that can be written as an integer or a fraction.
- *Axiom*: An axiom is a statement or proposition that is regarded as being accepted or self-evidently true.
    - <u>Example</u>: For real numbers a, b, c, it is true that a(b+c) = ab + ac.
- *Theorem*: A mathematical statement that has to be proved. Consider it to be absolute truth. Either someone else has already proved it for you and you can take it as true or you have to prove it.
    - Theorems are often "if/then" statements. When **using** a theorem, you have to show the" if" part works for your particular problem. The "then" part will come as a result of the "if." When **proving** a theorem, assume the "if" part and try to use that assumption and other things you know to prove the "then" part.
    - <u>Example</u>: If f is differentiable at c, then f is continuous at c.
- *Lemma*: A lemma is like a pre-theorem. It is a usually something that will help you in your proof of a more significant theorem.
- *Corollary*: Usually a "post-theorem." It often a direct consequence of the theorem you just proved.

<u>Types of Questions in a Proof Based Class:</u>

- *Compute*: This is the kind of question you are used to from your past math classes. You have a bunch of given information and you have to go through steps to get to the answer.
- *Show*: This is a less formal word for "prove." When you see this type of question, you have to ask yourself, "Does the professor or author of the book actually mean prove?" Usually the answer is "yes."
- *Verify:* This usually means you're given a theorem as well as a specific situation where it is applied and you have to show that the specific example follows the behavior predicted by the proof. These types of questions are usually more for your own understanding of the theorem.
- *Prove*: This means you have to take a given statement, and through mathematical logic, show that it is absolute truth. We will discuss more how to do this below.

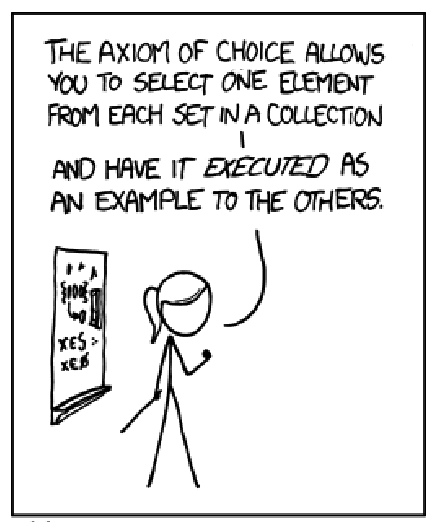
How to Write a Proof:
Writing a good proof is like writing a good essay: you need a topic sentence, body paragraph(s), and a concluding sentence. Each statement must be extremely clear so your reader can follow your logic. You need to be mindful of your intended audience. While the professor is actually the one reading your proof, you should write as though your reader is someone who doesn't understand why the theorem is true. You have to demonstrate that *you* know what is going on by writing as clearly as possible.

Structure of a Basic Proof for Beginners:

1. *Topic Sentence*: "We must prove" + WORDS.
    a. When I put WORDS in all caps I mean state the equation or the thing that you are trying to prove in words and not the equation itself. Say you are trying to prove a + c equals b. Instead of writing "We must prove a + c=b," write, "We must prove that a plus c equals b." If you write the answer in numerical form, you are more likely to accidentally use it as an assumption versus something you are trying to prove. This is a very common error for math students new to proofs.
    b. In addition to stating what you have to prove, be sure to translate this statement using relevant definitions. For example, in proving a number is even, you have to prove that there exists some integer j for which m = 2j. Your topic sentence for this proof would be "We must prove m is even, so we have to prove there exists some integer j with m = 2j."
2. *Body Paragraph*: "We know…"
    a. Begin the first sentence of your body paragraph(s) with the phrase "We know…" followed by an assumption or theorem you know to be true.
    b. Your goal is to start from somewhere you know to be 100% true and move to what you want to prove using other things you know to be true, like axioms or theorems.
3. *Concluding Sentence*: "We have proved..."
    a. Here you can write what you have proved in equation form, because you are done. For example, "Thus, we have proved 1=1." It's okay to write the equation because there isn't danger of being confused because you're done with the proof.

The Body Paragraph: Here are four basic types of proof often introduced in an introductory abstract mathematics class:

- *Direct Proofs:*
    - If you are trying to prove a + b = c, start from one side and try to use things you know to get to the other side of the equal sign. It is important that you not take the full equation a+b = c and manipulate it into another equality you know is true.
- *If and Only If:*
    - If and Only If proofs are for if and only if statements (often denoted "iff"); For example, the statement "P is an integer if and only if Q is an integer" is an if and only if statement. For this proof you can start by splitting your argument into 2 paragraphs. The first one should start by saying "Let's assume P is an integer. We must prove Q is an integer." After you've done that portion of your proof, you begin the second paragraph with "Let's assume Q is an integer. We must prove P is an integer." Be careful not to use any of the information in your first segment for your second segment as they are two different statements.



MY MATH TEACHER WAS A BIG BELIEVER IN PROOF BY INTIMIDATION.

- *Proof by Contradiction*
    - Proof by contradiction is a great tool when there are only two possibilities (or cases). For example, if you were given the problem: "Prove "P" is an integer," you could start your proof with "Assume P isn't an integer." Then use other things you know to try to reach a contradiction, i.e, that P has to be an integer.

- Another good tip-off for proof by contradiction is if the statement says that a certain kind of object doesn't exist. Using a proof by contradiction allows you to pretend that it does exist, and then play around with your assumptions until you find a conflict.

- *Proof by Induction*
  - A proof by induction can be extremely useful when you are trying to prove something is true for all n = 1, 2, 3, 4 …..The proof by induction has two parts: the base case and the induction case.
  - In a proof by induction, you start with a base case in which you let n equal your initial value in your sequence. The initial value is usually 1 or 2. In the base case, you must prove your theorem is true for that initial value.
  - In your induction step, you must generalize your findings and prove that your theorem is true for n = k+1. By proving it is true for n = k +1, you will have proved it is true for all n.
  - Example: For every positive integer n, prove 1 + 2 + 3 + … + n = n(n+1)/2

Tips for Tests:

- *When Preparing for the Test*
  - Make sure you really internalize your definitions and theorems--doing this over time means you'll know it and not have to cram before the test.
  - Make sure you can produce examples of various definitions and understand how you can use theorems to interpret the properties of these particular examples. In addition to examples, make sure you can produce counter-examples. Ask yourself: "What doesn't satisfy this definition?" and "What's an example where the hypothesis of the theorem is not satisfied and the conclusion is false?"
  - When studying on your own, do as many practice problems as possible. Make sure you can state and write all the necessary theorems and definitions
  - When studying with a group, practice stating theorems aloud, quiz each other, come up with practice scenarios.
- *When Taking the Test:*
  - If you have no idea about a proof what to do: play with examples, indicate that you know you know that you're wrong but are thinking about it, etc.
  - However, be careful to keep your "experimental work" separate from the formal proof. The phrase "for example" should not be part of your proof.

Style Points:

- *"Let":* If you want to select something random from a given set without having to chose an actual number, use the word "let." For example: "let a be any integer." If you were trying to prove something about all integers, you could algebraically play around with "a" for the rest of your proof, assuming it's an integer. This way you save yourself the enormous effort of proving whatever your theorem is for …-3, -2, -1, 0, 1, 2, 3…
- *"We":* Using the word "We" versus "I". For example, at the end of the proof, you might want to say "…thus, we have proved" versus "…thus, I have proved"
- *"Thus/Therefore/Hence/So/It Follows":* These are nice formal transition words that you can use from one step to another. For example, "We know that k is an even integer. **It follows/Hence/Thus/Therefore** that there exists some other integer j so that k = 2j.

Additional Resources:

- *How to Prove it* Matthias Beck and Ross Geoghegan
- *The Elements of Style* by E.B. White (for grammar and stylistic tips)
- *How to Write Proofs* http://zimmer.csufresno.edu/~larryc/proofs/proofs.html
- *Notes on Methods of Proof* http://www.math.csusb.edu/notes/proofs/pfnot/pfnot.html
- *Basic Proof Methods* http://homepages.math.uic.edu/~marker/math215/methods.pdf