The Poincaré conjecture and the shape of the universe

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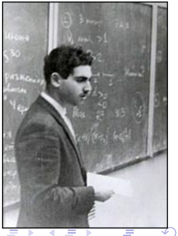
Poincaré conjecture: A century in two minutes

Henri Poincaré (1854-1912).
He invented algebraic topology.
It is a theory for distinguishing "shapes" of geometric objects through algebraic computations.

1904: Poincaré asks whether algebraic topology is powefull enough to characterize the shape of the 3-dimensional "hypersphere".
2002: Grigori Perelman (1966-) proves that the answer to Poincaré's question is YES ($\Rightarrow$ Fields medal.)

Theorem (Perelman, 2002)

Any closed manifold of dimension 3 whose fundamental group is trivial is homeomorphic to the hypersphere S^3.



What is the shape of the universe ?

Poincaré conjecture is related to the following problem:

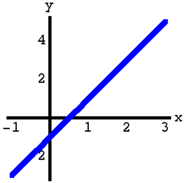
What "shape" can a three-dimensional "space" have ?

For example: what is the "shape" of the universe in which we live ?

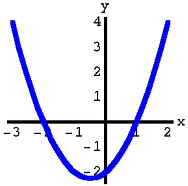
- The earth is *locally flat* but it is not an infinite plane.
- The earth has no boundary, so it is not a flat disk.
- The earth is a sphere: A finite surface which has no boundary.
- There is no reason why our universe should have the shape of the 3-dimensional Euclidean space.
  The universe could be finite without boundary.

1 equation with 2 unknowns $\Longrightarrow$ a curve

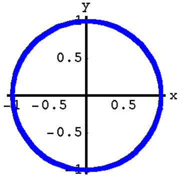
- $y = x - 1$
  S is a line



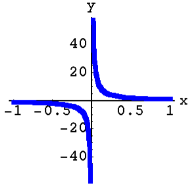
- $y = x^2 + x - 2$
  S is a parabola



- $x^2 + y^2 = 1$
  S is a circle

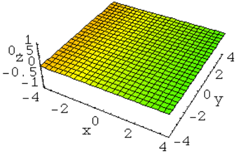


- $xy = 1$
  S is a hyperbola

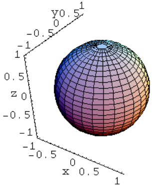


1 equation with 3 unkowns $\Longrightarrow$ a surface

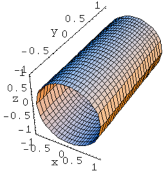
- $z = 0$
  S is an infinite plane



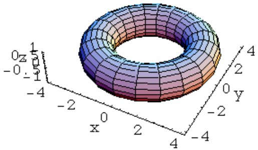
- $x^2 + y^2 + z^2 = 1$
  S is a sphere



- $x^2 + z^2 = 1$
  S is an infinite cylinder



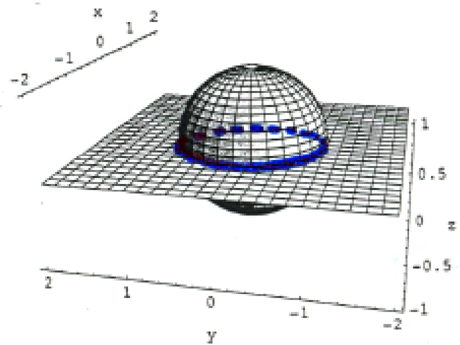
- $(8 + x^2 + y^2 + z^2)^2 = 36(x^2 + y^2)$
  S is a torus



2 equations with 3 unknowns $\Longrightarrow$ a curve

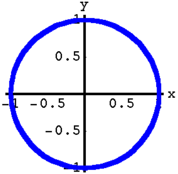
$$\begin{cases} x^2 + y^2 + z^2 & = & 1 \\ z & = & 0 \end{cases}$$

The set of solutions S is a circle.

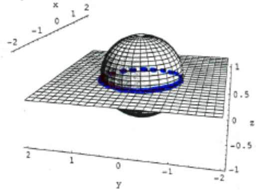


dimension = # unknowns − # equations

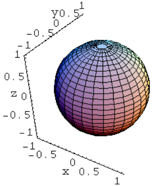
- $x^2 + y^2 = 1$
  The circle: dim $= 2 - 1 = 1$



- $\begin{cases} x^2 + y^2 + z^2 & = & 1 \\ z & = & 0 \end{cases}$
  The circle: dim $= 3 - 2 = 1$



- $x^2 + y^2 + z^2 = 1$
  The sphere: dim $= 3 - 1 = 2$



- $x^2 + y^2 + z^2 + t^2 = 1$
  dim $= 4 - 1 = 3$

$\boxed{\text{The hypersphere of dimension 3}}$

Manifold of dimension d

Definition

A manifold of dimension d or *d-manifold* is a geometric object that locally looks like a Euclidean space $\mathbf{R}^d$ of dimension d.

Examples:

- $d = 1$: curves (circle, line, hyperbola, ...)
- $d = 2$: surfaces (sphere, plane, torus, ...)
- $d = 3$: hypersphere, "the universe",...

In this talk we will only consider d-manifolds that are:

- "finite" (not the line, or the parabola, or the plane)
- connected, so it is in one piece (not the hyperbola)
- without boundary (not a disk, nor a segment)

Restatement of Poincaré's problem

The question asked by Poincaré:

> *Give an easy criterion that enables us to recognize that a given* 3*-manifold has the same shape as the* 3*-dimensional hypersphere.*

More generally Poincaré was looking for a classification of possible shapes of all 3-manifolds.

Let us begin with an easier problem:

The classification of shapes of manifolds of dimension 1 and 2.

Classification of the shapes of 1-manifolds, i.e. of curves

Theorem (Classification of 1-manifolds)

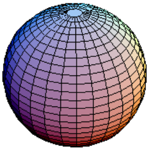
Any curve (finite, connected, without boundary) has the shape of a circle.

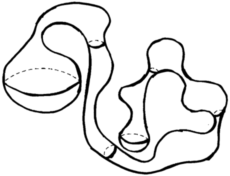
These three figures have the same shape, i.e. they are homeomorphic:

- One can deform one into another ("rubber geometry")
- A tiny observer living on the curves, could not distinguish them.

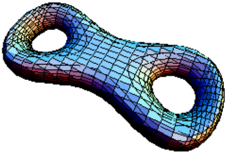
Examples of 2-manifolds, i.e. of surfaces

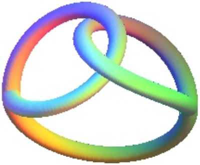
- Shape of a sphere



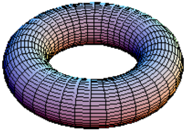


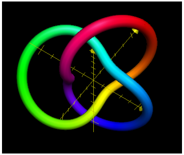
- Shape of a torus with two holes



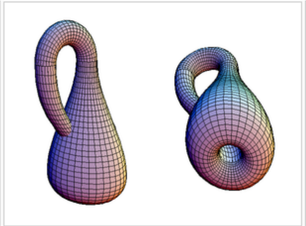


- Shape of a torus





- Shape of a Klein bottle



Deux vues de la bouteille de Klein dans un espace à trois dimensions.

Another Klein bottle

Classification of the shapes of 2-manifolds, i.e. of surfaces

Theorem (Classification of surfaces)

Any surface (finite, connected, without boundary) has the same shape as a torus with $g = 0, 1, 2, 3, \dots$ holes or as the unorientable version of a g-torus.

Explanations:

- By definition the torus with 0 holes is the sphere.
- The Klein bottle is the unorientable version of the torus with one hole.
- Each torus with g holes has exactly one unorientable "sister."

The following is a complete list of surfaces (up to homeomorphism):

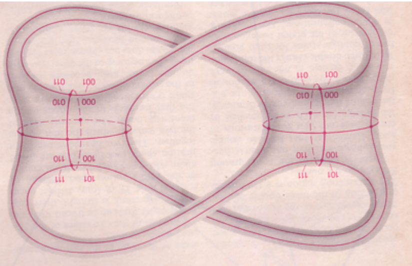
$+$ unorientable versions.

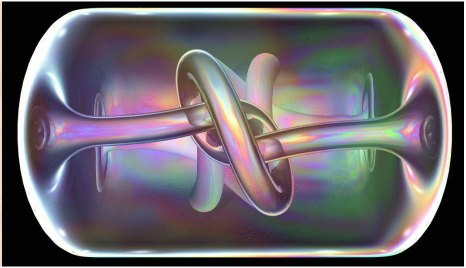
How do we recognize a given surface in this list?

The classification theorem of surfaces tells us that in order to recognize the shape of a surface it is enough to:

1. Know its genus g, that is the number of holes, and
2. Know whether it is orientable or not.

How do we proceed in practice to determine g ?
For example: how many holes do the following two surfaces have ?





Poincaré's criterion for recognizing whether a surface is a sphere

The sphere is the only surface that fullfills the "Poincaré's criterion":

$$\boxed{\text{every closed loop can be shrunk to a point.}}$$

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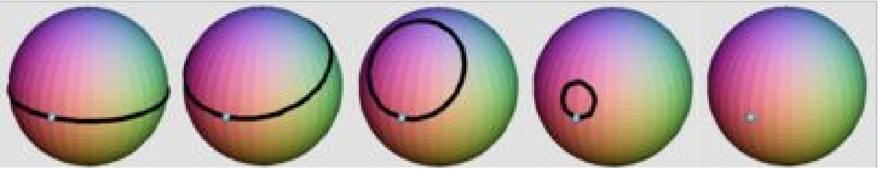


Poincaré conjecture

The hypersphere is the only 3-manifold for which every closed loop can be shrunk to a point.

Digression: Curvature and geometry.

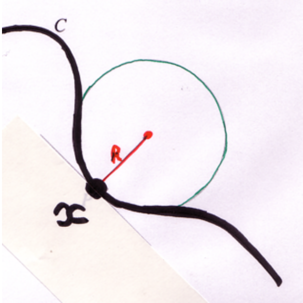
Topology Study of global shapes. “Rubber geometry.”

Geometry Study of objets on which one can measure lengths and angles. Notion of curvature.

Curvature of a plane curve

The curvature of a plane curve C at a point $x \in C$ is the real number

$$k(x) = \pm \frac{1}{R}$$



- R = radius of the best approximating circle to the curve;
- The sign $\pm$ depends on the position of the centre of the circle.

Examples:

- For a line: $k(x) =$

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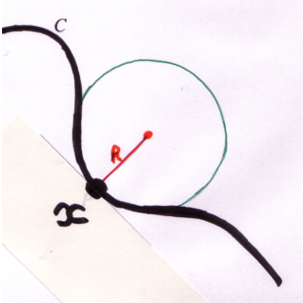

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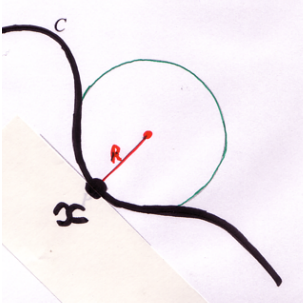

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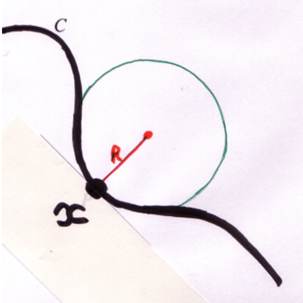

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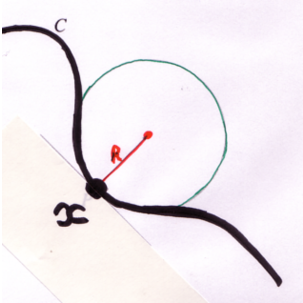

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Examples:

- For a line: $k(x) = 0$
- For a circle of radius R: $k(x) = \pm 1/R$
- For a sine curve : $k(x)$

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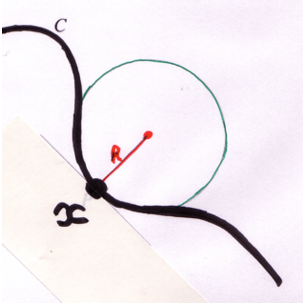

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- The sign $\pm$ depends on the position of the centre of the circle.

Examples:

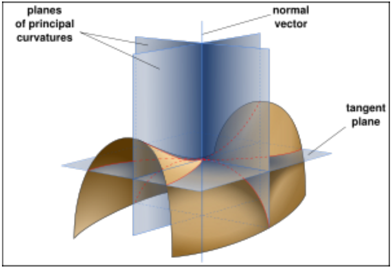
- For a line: $k(x) = 0$
- For a circle of radius R: $k(x) = \pm 1/R$
- For a sine curve : $k(x)$ oscillates between -1 et 1

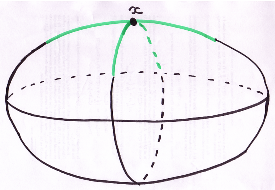
Gaussian curvature of a surface

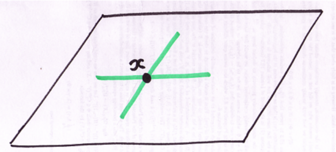
The Gaussian curvature of a surface at a point x is the real number

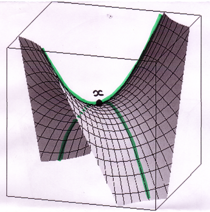
$$K(x) \ := \ k_{\min}(x) \cdot k_{\max}(x)$$

where $k_{\min}$ is the minimal curvature of a curve on the surface passing through x inside a normal plane to the surface, and analogously $k_{\max}$ is the maximal curvature of such a curve.









$K(x) = \frac{1}{R_1} \cdot \frac{1}{R_2} > 0$ $\qquad$ $K(x) = 0 \cdot 0 = 0$ $\qquad$ $K(x) = \frac{1}{R_1} \cdot \frac{-1}{R_2} < 0$

The Gaussian curvature K is invariant under isometries

A priori it seems that the Gaussian curvature $K(x)$ depends on the way the surface is embedded in the 3-dimensional euclidean space $\mathbf{R}^3$. Actually it does not: It only depends on the intrinsic geometry of the surface, i.e. on the way lengths are measured on the surface.

Theorema egregium (Gauss, 1828)

If a map $f: S_1 \to S_2$ between two surfaces S_1 and S_2 embedded in $\mathbf{R}^3$ preserves inside lengths then it also preserves Gaussian curvature:

$$K(f(x)) = K(x).$$

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The pizza corollary

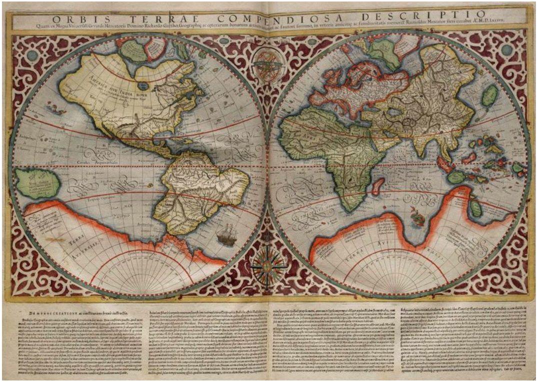
If a surface S is isometric to a subset of the plane then its Gaussian curvature is zero everywhere:

$$k_{\min}(x) \cdot k_{\max}(x) = 0.$$

The round sphere is not isometric to a portion of the plane

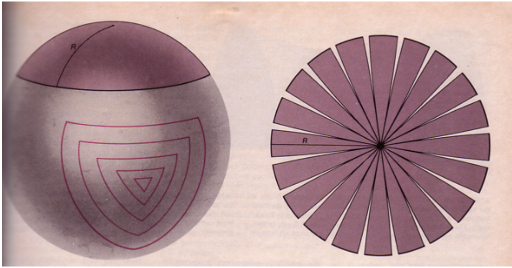
The cartographer's nightmare corollary

It is impossible to draw a map representing the surface of the earth on a plane in such a way that distances are preserved (or scaled).



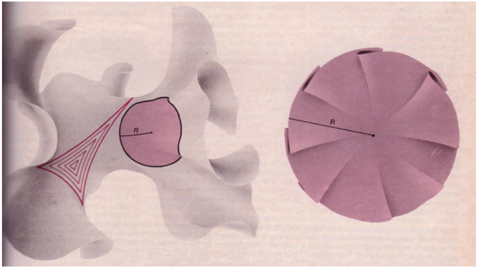
Mercator Planisphere (1512 – 1594)

Positive and negative Gaussian curvatures



$$\boxed{K > 0}$$

- The sum of the anges of a triangle $> 180^\circ$
- The area of a disk of radius R is $< 2\pi R^2$
- Spherical geometry



$$\boxed{K < 0}$$

- The sum of the anges of a triangle $< 180^\circ$
- The area of a disk of radius R is $> 2\pi R^2$
- Hyperbolic geometry

Curvature of manifolds of dimension $d \geq 3$

Gaussian curvature can be generalized in dimension ≥ 3:

$$\boxed{\text{The Ricci curvature.}}$$

This curvature measures how much the geometry of the manifold differs from Euclidean geometry.

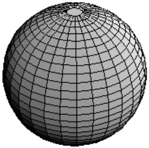
In general relativity, the Ricci curvature is the main ingredient to quantify how much mass and energy deform the space-time geometry.

Link between curvature and topology

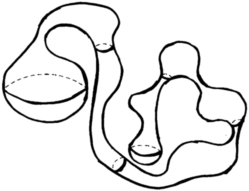
Difference between topology and geometry.

- Topology is the study of “global shapes” of geometric objects. Objects are flabby.
- Geometry is the study of properties related to the “length” on geometric objects.
  Objects are more rigid: we cannot stretch them.
- Curvature is an invariant of geometry (theorema egregium)

Is there a link between curvature and the global shape of a manifold ?



$K(x) = 1/R^2$ is constant



$K(x)$ is very variable

At first sight: No link between curvature and topology of the sphere.

Curvature constraints topology

$\overline{K}$:= mean value of the Gaussian curvature of a surface
$\mathrm{area}(S)$:= area of the surface
g := number of holes (or genus) of the surface.

Theorem (Gauss-Bonnet)

$$\overline{K} \cdot \mathrm{area}(S) \; = \; 4\pi(1-g)$$

Corollary

For a surface S with a homogeneous geometry, we have

- if $K(x) > 0$ then S is a sphere;
- if $K(x) = 0$ then S is a torus (with one hole);
- if $K(x) < 0$ then S is a torus with $g \geq 2$ holes;

or their non-orientable variants.

Conclusion: for a homogeneous surface (that is, with constant curvature), the sign of the curvature partially determines the topology.

Poincaré conjecture for homogeneous 3-manifolds

Theorem

A manifold of dimension d whose curvature at any point is constant and that satisfies Poincaré criterion (any closed loop can be shrunk to a point) is isometric to the round hypersphere of dimension d.

- This theorem implies that Poincaré conjecture is true for manifolds with constant curvature, i.e. with homogeneous geometry.
- This theorem was known for a long time and is not difficult to prove.
- Idea of the proof:
  - A space with constant curvature is locally isometric to the round hypersphere.
  - Poincaré criterion implies that when we glue together the local patches isometric to portions of hypersphere we get the genuine hypersphere and not some variant like its unorientable version.

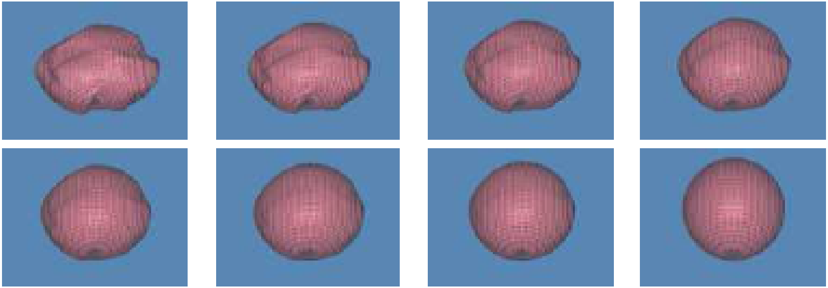
Perelman's proof of the Poincaré conjecture

Theorem (Perelman, 2002)

Any 3-manifold in which any closed loop can be deformed into a point is homeomorphic to the hypersphere of dimension 3.

The proof follows a program initiated by Richard Hamilton (1982)

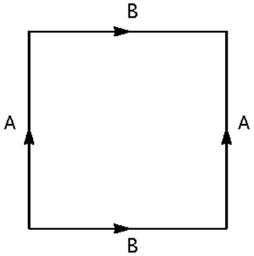
1. Equip the manifold with an arbitrary geometry.
2. Let the geometry of this 3-manifold "cool down".
3. After some time the curvature becomes constant.
4. Therefore it is the hypersphere . QED

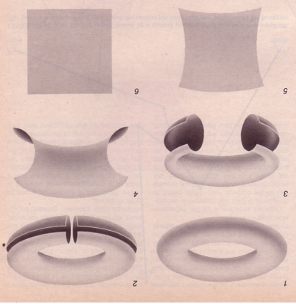


Examples of 3-manifolds other than the hypersphere.

Construction of surfaces from polygons

- The torus is obtained by gluing the opposite sides of a square:

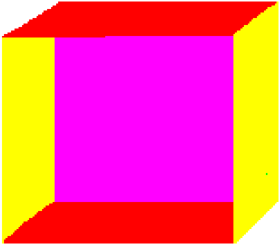


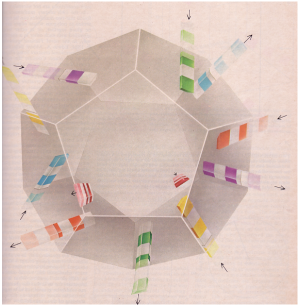


- By gluing the opposite sides of an octagon one obtains a torus with two holes.
- Any surface is obtained by gluing together pairs of sides of some polygon and conversely.

Construction of 3-manifolds from polyhedra

- The "hypertorus" is the 3-manifold obtained from a plain cube by gluing together the oppposite faces.
- Poincaré invented a space obtained by gluing together the opposite pentagonal faces of a plain dodecahedron.

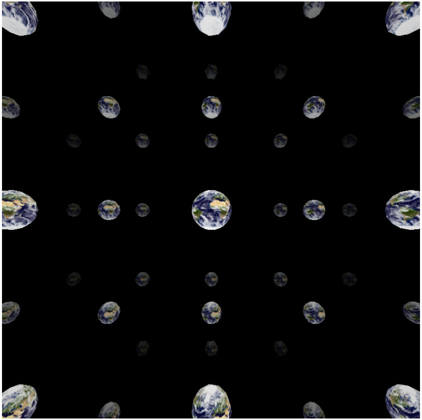


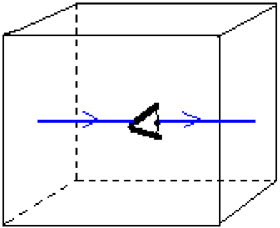


Any 3-manifold is obtained from a plain polyhedron whose faces are identified in pairs.

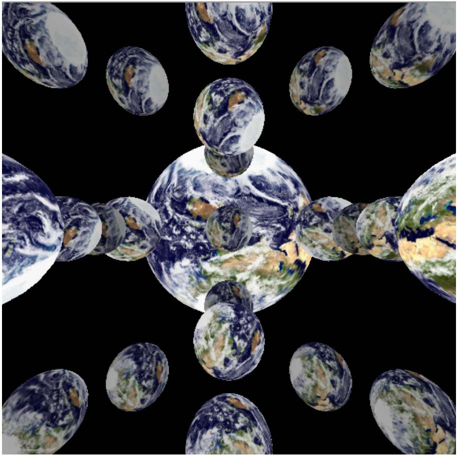
If the universe was a hypertorus

Imagine that the universe is a hypertorus (i.e. the cube with the parallel faces identified) with a homogeneous geometry

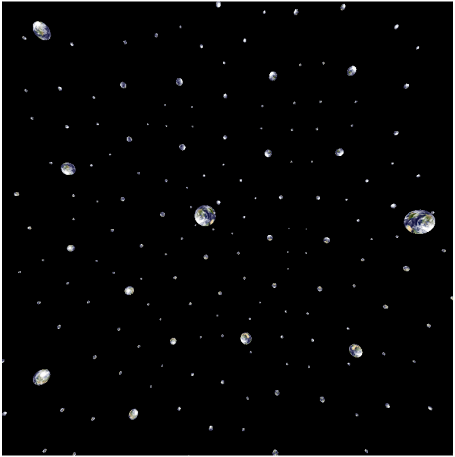




Two dodecahedric universes



The dodecahedric 3-manifold of
Poincaré
Curvature > 0



The dodecahedric 3-manifold of
Seifert-Weibel
Curvature < 0

Conclusion

Perelman proved much more: It appears that he has completely characterized all 3-manifolds by their geometry, achieving a research program initiated by William Thurston in 1982.
The classification of 3-manifolds may soon be completed !!!

And in dimension > 3?

- Dimension $d = 4$. The analog of the Poincaré conjecture was proved in 1982 (M. Freedman).
  But there are still mysterious phenomena in dimension 4.
- Dimensions $d \geq 5$: the classification of d-manifolds has been well understood since the end of the 1960's !
  The analog of the Poincaré conjecture was proved in 1961 (S. Smale).

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And the shape of the universe ?
Until recently, it had been speculated that it could be the dodecahedric Poincaré space!!! It seems that very recently this speculation was ruled out.

A few references

- Jeffrey Weeks "The shape of space".
  An introductory book for the notion of shape and for the shape of the universe.
- W. Thurston et J. Weeks "The Mathematics of Three-dimensional Manifolds" . Scientific American, vol. 251, July 1984, p. 108.
  A very readable paper on the classification project for 3-manifolds based on their geometry.
- J. Weeks "The Poincaré dodecahedral space and the mystery of the missing fluctuations" Notices A.M.S., Vol. 51 (2004), pp. 610-619.
  An article for mathematicians showing some experimental evidences that the universe could be the dodecahedric Poincaré space.
- W. Thurston "Three-dimensional geometry and topology"
  An excellent book in which a few chapters are readable by a motivated student in mathematics.
- http://www.geometrygames.org/CurvedSpaces
  "A flight simulator" in 3-manifolds with homogeneous geometries.