Integrals don't have anything to do with graph theory . . . do they?

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Wellesley College — 16 March 2009

Outline

Euler's Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

- converges for $0 < x < \infty$
- extends to $\mathbb{C}$; meromorphic with poles $\{0, -1, -2, -3, \ldots\}$
- $\Gamma(1) = -e^{-t}\Big|_0^\infty = 1$
- $\Gamma(1/2) = \sqrt{\pi}$
- $\Gamma(1/3) = ??$ $\quad \Gamma(1/4) = ??$

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Gamma Function Recurrence

Integrate by parts...

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$$

$$= -t^x e^{-t}\Big|_{t=0}^{\infty} + \int_0^\infty x\,t^{x-1} e^{-t}\,dt$$

$$= 0 + x\,\Gamma(x)$$

Corollary

$$\Gamma(n+1) = n!$$

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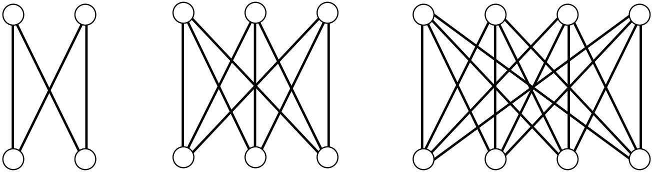
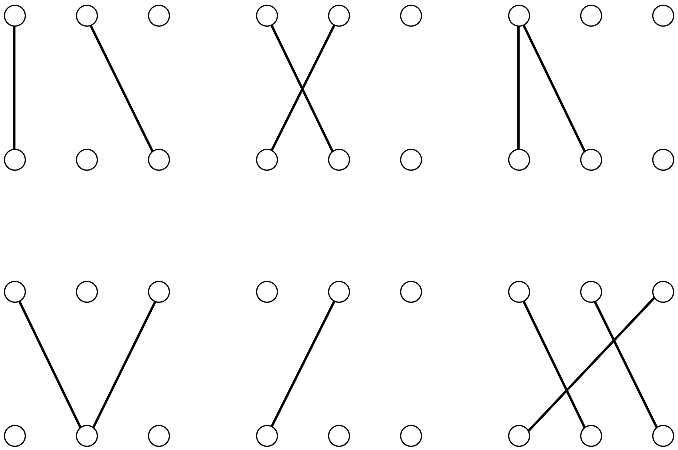
Complete Bipartite Graphs



$K_{2,2}$ $K_{3,3}$ $K_{4,4}$

Skip to bottom line

Matchings in $K_{3,3}$



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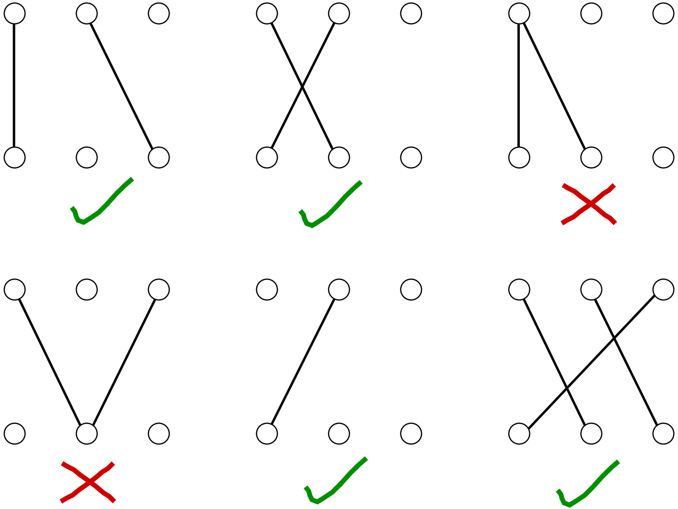
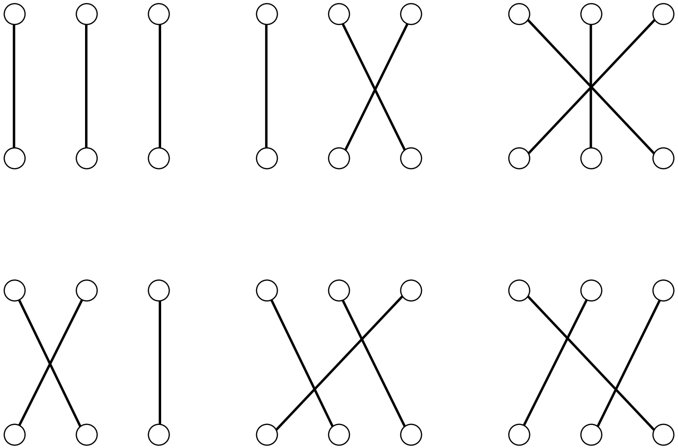
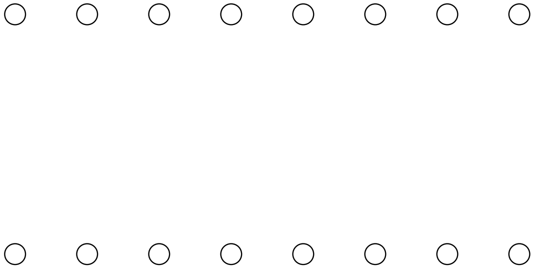


Perfect Matchings in $K_{3,3}$



Counting Perfect Matchings in $K_{n,n}$



Counting Perfect Matchings in $K_{n,n}$

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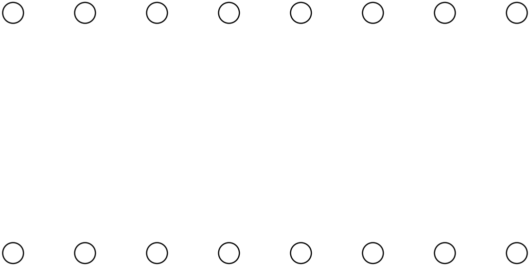

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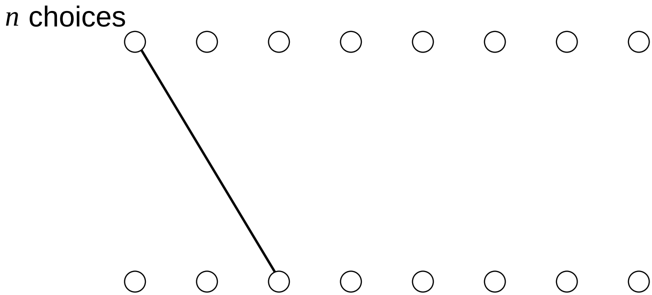

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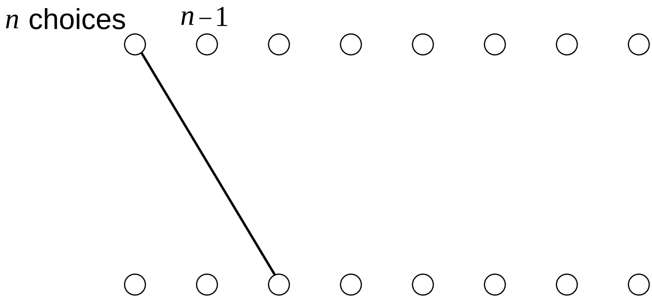

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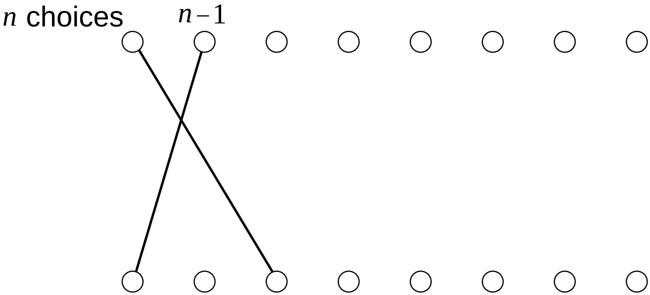

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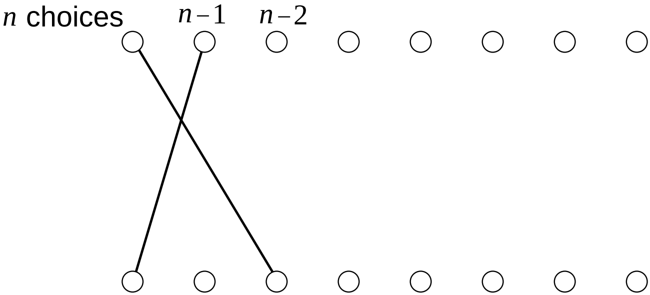

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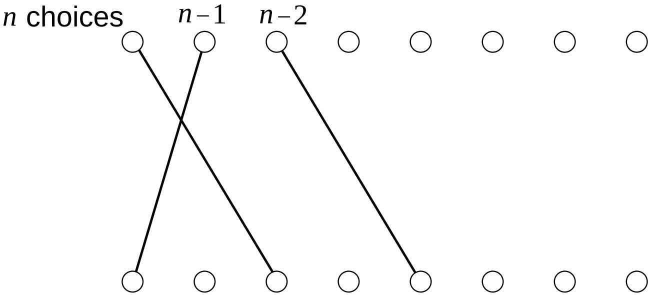

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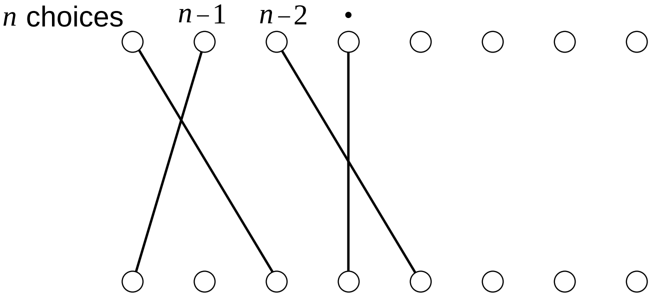

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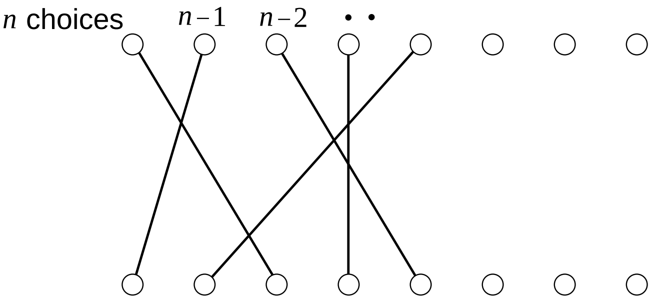
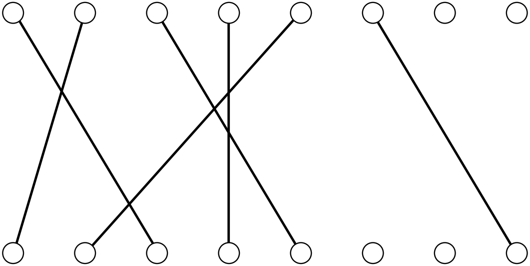


Counting Perfect Matchings in $K_{n,n}$

n choices $\quad n-1 \quad n-2 \quad \cdots$



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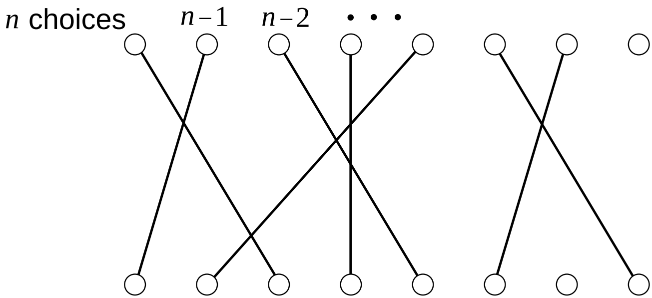

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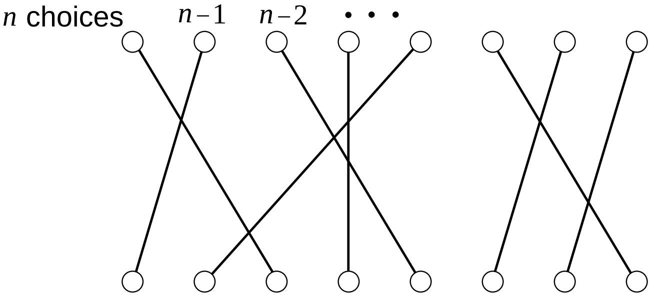

Counting Perfect Matchings in $K_{n,n}$: Bottom Line

Proposition

$$\Xi(K_{n,n}) = n!$$

Proposition

$$\Xi(K_{n,n}) = \int_0^\infty t^n e^{-t}\, dt$$

Counting Perfect Matchings in $K_{n,n}$: Bottom Line

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Tinkering with $K_{n,n}$

$$\Xi(K_{n,n}) = \int_0^\infty t^n e^{-t}\,dt$$

Question

If we replace $K_{n,n}$ by a subgraph $G \subseteq K_{n,n}$, then how does the integrand change?

$$\Xi(G) = \int_0^\infty \;?\; dt$$

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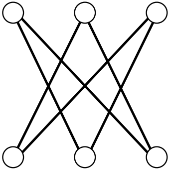

G : spanning subgraph of $K_{n,n}$

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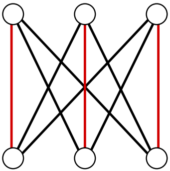


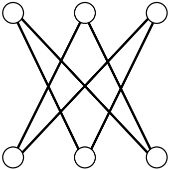
G : spanning subgraph of $K_{n,n}$

$\widetilde{G}$: bipartite complement of G

Notation

Matchings with a fixed number of edges

$\mu_G(k)$ = number of matchings of G containing exactly k edges



k	0	1	2	3
$\mu_G(k)$	1	6	9	2

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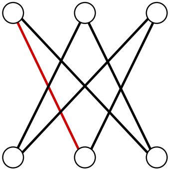

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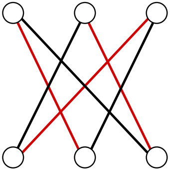

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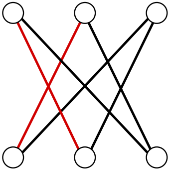

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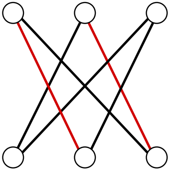

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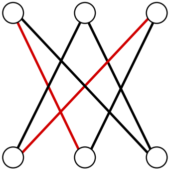

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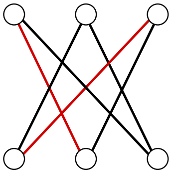

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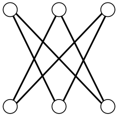
The Rook Polynomial

An example



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$$\rho_G(t) = +1t^3 \quad -6t^2 \quad +9t \quad -2$$

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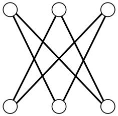

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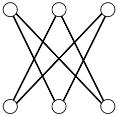

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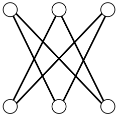

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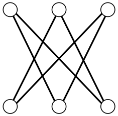

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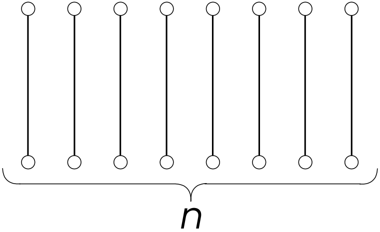
Definition (rook polynomial of spanning $G \subseteq K_{n,n}$)

$$\rho_G(t) = \sum_{k=0}^{n} (-1)^k \mu_G(k) t^{n-k}$$

The Rook Polynomial

Another example



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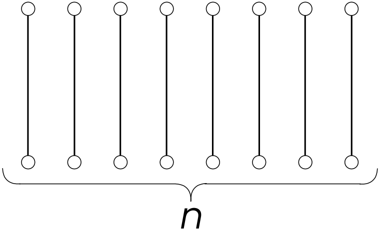

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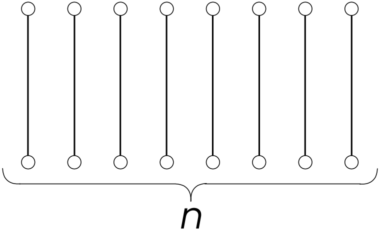

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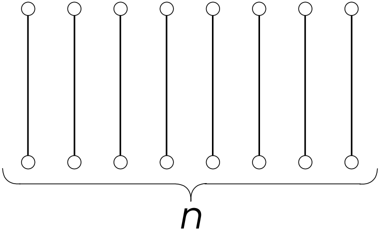

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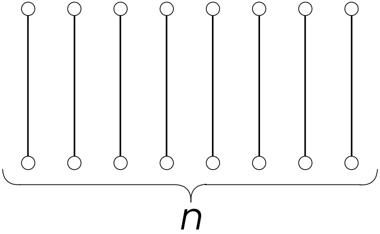

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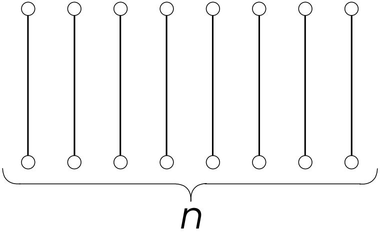

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Tinkering with $K_{n,n}$

$$\Xi(K_{n,n}) = \int_0^\infty t^n e^{-t}\, dt$$

Question

If we replace $K_{n,n}$ by a subgraph $G \subseteq K_{n,n}$, then how does the integrand change?

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Counting with Integrals

Proposition

$$\Xi(K_{n,n}) = \int_0^\infty t^n e^{-t}\, dt.$$

Theorem (Joni & Rota, 1980; Godsil, 1981)

If G is a spanning subgraph of $K_{n,n}$, then

$$\Xi(G) = \int_0^\infty \rho_{\widetilde{G}}(t) e^{-t}\, dt.$$

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See proof

Skip proof

Proof of Main Result

Set-up

- $\mathcal{X} \;=\;$ set of perfect matchings of $K_{n,n}$
- $E(\widetilde{G}) \;=\; \{1, 2, \ldots, m\}$ with $m \geq 0$
- for $i \in E(\widetilde{G})$: $A_i = \{M \in \mathcal{X} : M \text{ contains edge } i\}$
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Therefore $$\Xi(G) \;=\; \left| \mathcal{X} - \bigcup_{i=1}^{m} A_i \right|$$

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$$\Xi(G) = \left| \mathcal{X} - \bigcup_{i=1}^{m} A_i \right|$$

$$\Xi(G) = |\mathcal{X}| - \sum_{i=1}^{m} |A_i| + \sum_{1 \le i < j \le m} |A_i \cap A_j| - + \cdots$$
$$\cdots + (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| + \cdots$$

Notice: $A_{i_1} \cap \cdots \cap A_{i_k} = \varnothing$ unless $\{i_1, \ldots, i_k\}$ is a matching in $\widetilde{G}$.

$\{i_1, i_2, \ldots, i_k\}$ matching in $\widetilde{G} \implies |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$

There are $\mu_{\widetilde{G}}(k)$ such matchings in $\widetilde{G}$.

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There are $\mu_{\widetilde{G}}(k)$ such matchings in $\widetilde{G}$.

Therefore
$$\Xi(G) = \sum_{k \ge 0} (-1)^k \mu_{\widetilde{G}}(k)(n-k)!$$

Proof of Main Result

Conclusion

$$\Xi(G) = \sum_{k=0}^{n}(-1)^k \mu_{\widetilde{G}}(k)(n-k)!$$

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To Appendix
Wrap-up

Derangements: introduction

Definition

A derangement σ of a set S is a permutation of S admitting no fixed points; i.e., $\sigma : S \to S$ is a bijection such that $\sigma(x) \neq x$ for each $x \in S$.

Notation

$$d_n = \text{number of derangements of } \{1, 2, \ldots, n\}$$

Example

n	d_n	derangements of $\{1, 2, \ldots, n\}$
1	0	$\varnothing$
2	1	$\{21\}$
3	2	$\{231, 312\}$
4	9	$\{2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321\}$
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Question

What is the value of d_n?

Answer

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Proof technique.

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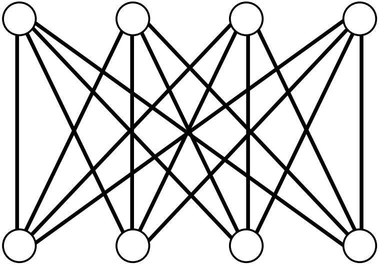

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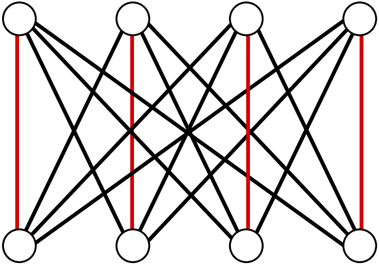

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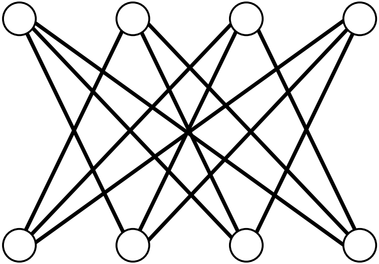

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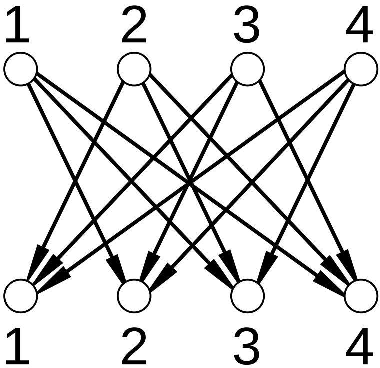

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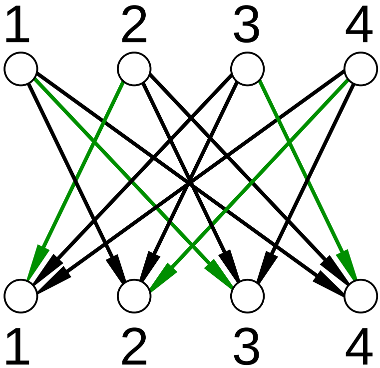

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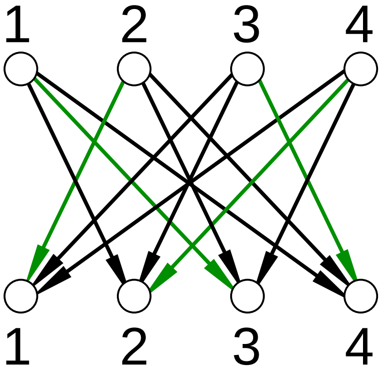

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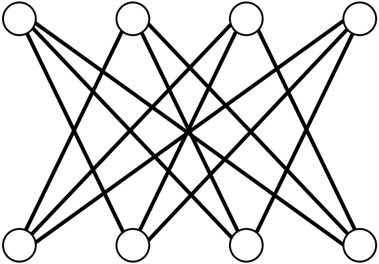

$$d_n = \Xi(K_{n,n} - M)$$

Recall the Main Result

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$$E_n = \int_0^1 (t-1)^n e^{-t}\, dt$$

$e^{-t} < 1$ on the interval $(0,1)$

$$|E_n| \leq \int_0^1 \left|(t-1)^n e^{-t}\right| dt < \int_0^1 (1-t)^n\, dt = \frac{1}{n+1}$$

$|E_n| < \frac{1}{2}$ for each $n \geq 1$

Derangements: denouement

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To Appendix
Wrap-up
Back to proof

Take-Home Messages

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- Ties can be fruitful, not merely beautiful.

fin

applause (optional)

More Counting with Integrals

Proposition

$$\Xi(K_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^n e^{-t^2/2}\, dt.$$

Theorem (Godsil, 1981)

If G is a spanning subgraph of K_n, then

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Rook polynomials

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Other combinatorially interesting sequences take the form

$$\int_\Omega t^n \, d\mu$$

for some measure μ and space Ω.

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Further Reading

Books

- E. Artin.
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  John Wiley & Sons, Inc., New York, 1958.

Further Reading

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Submitted for publication, 2009.

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A vector space analog of permutations with restricted position.
J. Combin. Theory Ser. A **29** (1980), 59–73.

Really *fin*

more applause (take it home)