III. Functions and straight lines

A. Functions

1. Definition   A function defined on a certain set of real numbers D (called the domain of the function) is a rule that associates to each element of D a real number.

For example, suppose f is the function that assigns to each real number the number obtained by doubling and adding 1. Functional notation is a useful shorthand for describing this rule. We write

$$f(x) = 2x + 1 \quad \text{for every real number } x$$

(or just $f(x) = 2x + 1$, with the domain tacitly understood). Thus we're using "$f(x)$" as a name for the number f assigns to x, which we call the value of f at x. To find the value of f at $*$, whatever $*$ is, we use the rule:

$$\text{value of } f \text{ at } * = f(*) = 2\cdot(*) + 1,$$

e.g. $f(3) = 2\cdot3 + 1 = 7$ $\qquad f(x+2) = 2(x+2) + 1 = 2x + 5$

$f(0) = 2\cdot0 + 1 = 1$ $\qquad f(x+h) = 2(x+h) + 1$

$f(\pi) = 2\pi + 1$ $\qquad f(1/z) = 2/z + 1$

2. Domain of a function   The domain of f is the set of all x for which $f(x)$ is defined. If the function is given by a formula and the domain is not explicitly specified, the domain is understood to be the largest possible set of numbers for which the formula makes sense.

Examples:

$f(x) = 2x + 1$. Domain = the set of all real numbers = {all x}.

$g(x) = \sqrt{x}$. Domain = set of all nonnegative numbers = $\{x \geq 0\}$.

(remember that $\sqrt{}$ always denotes positive square root)

$h(x) = \dfrac{1}{x-3}$. Domain = set of all numbers except 3 = $\{x \neq 3\}$.

3. Composition of functions   If f and g are functions, we may form their composition $g \circ f$, defined by $(g \circ f)(x) = g(f(x))$. This is a new function whose rule is "do f, then g", and whose domain consists of all real numbers x such that f is defined at x (x is in the domain of f) and g is defined at $f(x)$ ($f(x)$ is in the domain of g).

Examples:

(1) $f(x) = 2x + 1$, $g(x) = \sqrt{x}$. $(g \circ f)(x) = g(f(x)) = g(2x + 1) = \sqrt{2x+1}$.

f is defined for all x but $g(2x + 1)$ is defined only when $2x + 1$ is nonnegative, so the domain of $g \circ f$ is $\{ x \geq -1/2 \}$.

Note that in general, $g \circ f \neq f \circ g$. In Example 1, $(f \circ g)(x) = f(\sqrt{x})$ $= 2\sqrt{x} + 1$, which is not the same function as $\sqrt{2x+1}$, and

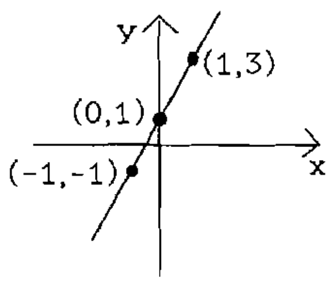
$f \circ g$ has domain $\{ x \geq 0 \}$.

(2) $f(x) = x^2$, $g(x) = \sqrt{x}$. $(g \circ f)(x) = g(x^2) = \sqrt{x^2} = x$; domain is { all x } because f is defined for all x and $g(x^2)$ is defined whenever $x^2 \geq 0$, which is always. By contrast, $(f \circ g)(x) = f(\sqrt{x})$ $= (\sqrt{x})^2 = x$; domain is $\{ x \geq 0 \}$ because g is only defined for such x , even though the final expression $(f \circ g)(x) = x$ appears to be defined for all x . So here again, $f \circ g \neq g \circ f$.

4. <u>Graph of a function</u> The graph of f is the set of all ordered pairs (x,y) such that x is in the domain of f and y is equal to $f(x)$. In other words it consists of all points of the form $(x, f(x))$ for x in the domain of f . The graph of f is plotted as a curve $y = f(x)$ in the (x,y)-coordinate plane.

Example: For the function $f(x) = 2x + 1$, the graph consists of all (x,y) satisfying $y = 2x + 1$.

This is a straight line.



x	y = f(x)
0	1
1	3
-1	-1

Unless the formula for the function fits some familiar form, as this one does, the graph may be hard to sketch accurately. It is always possible, though, to plot several points to get a rough idea.

<u>Graphing suggestions</u>

(a) Locate the intercepts.

The x-intercept is found by solving $y = f(x) = 0$ for x .

The y-intercept is found by setting $x = 0$: $y = f(0)$.

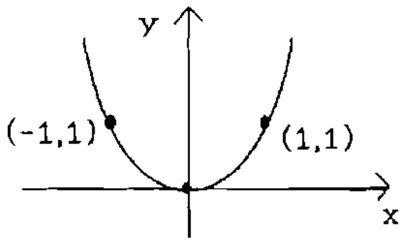
In other words the graph hits the x-axis where $y = 0$ and the y-axis where $x = 0$. In the example above, $y = f(x) = 2x + 1 = 0$ when $x = -½$ (x-intercept) and $y = f(0) = 1$ is the y-intercept. Also notice intercepts in the examples below.

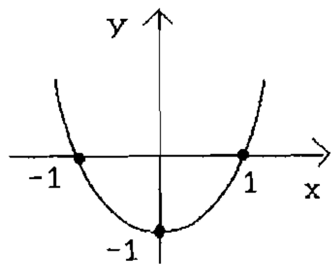
(b) Determine the domain to know what part of the x-axis the graph will lie over, e.g. the graph of $f(x) = \sqrt{x}$ has no portion lying to the left of the y-axis, and the graph of $g(x) = 1/x$ does not meet the y-axis because x cannot equal 0 .

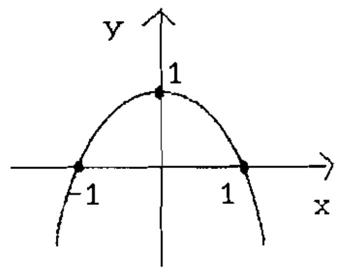
(c) Adding a constant to a function moves its graph up (if constant is

positive) or down (if negative) by that amount.
(d) Multiplying a function by -1 flips its graph about the x-axis.

Examples: $f(x) = x^2$ $g(x) = x^2 - 1$ $h(x) = -x^2 + 1$

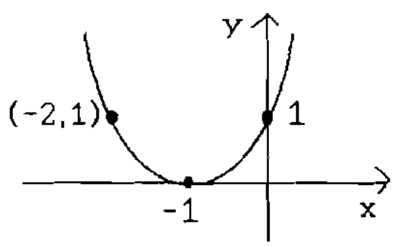


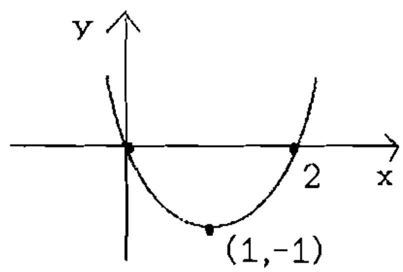




(e) Adding a constant to x itself, i.e. forming $f(x+c)$, moves the graph c units to the left (if c positive) or right (if c negative).

Examples: $p(x) = (x+1)^2$ $q(x) = (x-1)^2 - 1$





<u>Exercises **III A**</u>

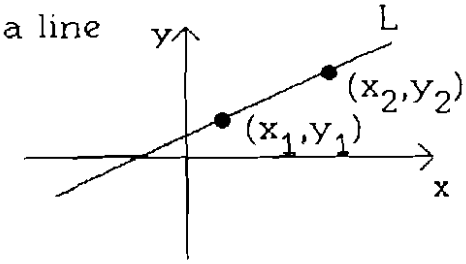
1. If $f(x) = 3x + 2$ and $g(x) = x + (1/x)$, find: (a) $f(2)$ (b) $f(x+3)$
(c) $g(½)$ (d) $g(x^2)$ (e) $g(0)$ (h) $(g \circ f)(x)$ (i) $(g \circ f)(-1)$ (j) $(f \circ g)(-1)$
2. If $h(z) = \frac{2}{3}(z+1)^{3/2}$, find $h(3) - h(0)$.
3. Find the domains of (a) $f(x) = \sqrt{x-2}$ (b) $g(x) = \dfrac{1}{\sqrt{x-2}}$.
(c) $g \circ f$ (d) $f \circ g$ with f and g as in Exercise 1.

Sketch the graphs of the following functions.

4. $f(x) = 2x$ 5. $g(x) = 2(x+3)$ 6. $h(x) = 3 - x^2$
7. $f(x) = \frac{1}{x}$ 8. $q(x) = -\frac{1}{x} + 2$ 9. $r(x) = \dfrac{1}{x-2}$

B. Lines and their equations

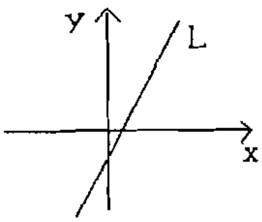
1. Slope of a line



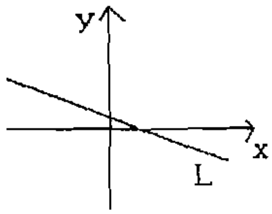
The slope of L is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{"rise"}}{\text{"run"}}$$

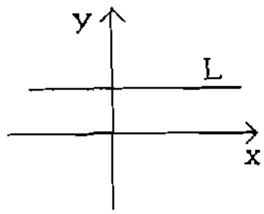
Positive slope (points up to right)



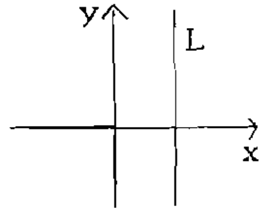
Negative slope (points down to right)



Zero slope (horizontal)

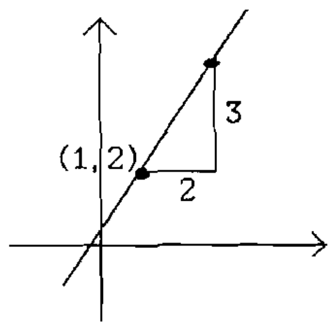


Undefined slope (vertical)

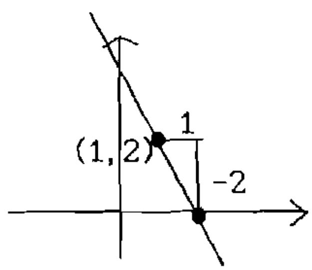


Examples (a) Sketch the line through (1,2) with slope 3/2 .

Locate (1,2) and mark it. Move by making a "run" of 2 and a "rise" of 3 . Mark new point. Join points by a line.



(b) The line through (1,2) with slope −2 . (Write slope as −2/1 ; run = 1 , rise = −2 .)



2. Point-slope form for equation of a line.
Line through (x_0,y_0) with slope m : A point will be on this line if and only if slope from (x_0,y_0) to (x,y) equals m . This condition can be written $\dfrac{y-y_0}{x-x_0} = m$ or $\boxed{y - y_0 = m(x - x_0)}$

For example, the line through $(1,2)$ with slope $3/2$ has equation

$$y - 2 = \frac{3}{2}(x-1) \qquad \text{or} \qquad y = \frac{3}{2}x + \frac{1}{2} .$$

3. Slope-intercept form for equation of a line.
   Line with slope m and y-intercept b (i.e. passes through the point $(0,b)$) is, from above,

$$y - b = m(x-0) \qquad \text{or} \qquad \boxed{y = mx + b}$$

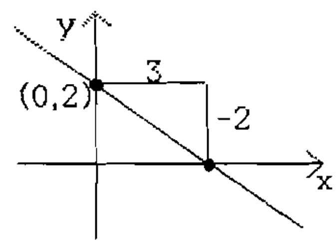
For example, $y = 2x + 3$ has slope 2 and y-intercept 3 .
This form is especially useful for graphing.

Example: Graph $2x + 3y - 6 = 0$.

Rewritten in slope-intercept form:

$$y = -\frac{2}{3}x + 2 .$$

Locate intercept $(0,2)$ and sketch line with slope $-2/3$.



4. Parallel and perpendicular lines.
   Two lines are parallel if they have equal slopes or both are vertical.
   Two lines are perpendicular if their slopes, m_1 and m_2 , are negative reciprocals of each other ($m_1 = -1/m_2$ or $m_1 m_2 = -1$) or one is horizontal and the other is vertical.

Examples $y = 2x + 3$ and $y = 2x + 5$ are parallel.
$y = 2x + 3$ and $y = -(1/2)x + 6$ are perpendicular.
$y = 3$ and $x = 2$ are perpendicular.
$y = 3$ and $y = 5$ are parallel.

The equation of the line through $(2,3)$ parallel to $y = 4x + 11$ is
$$y-3 = 4(x-2) .$$
The line through $(2,3)$ perpendicular to $y = 4x + 11$ has equation
$$y-3 = -(1/4)(x-2) .$$

Exercises III B

1. Rewrite in slope-intercept form and sketch.
   (a) $x - 2y + 2 = 0$ (b) $3x - 2y + 6 = 0$ (c) $2x + 3y - 1 = 0$

Write the equation for each of the following.

2. The line through $(4,1)$ with slope 2 .
3. The line through $(-1,2)$ and $(3,1)$. (First find the slope.)
4. The line through $(4,7)$ parallel to $y = 3x + 7$.
5. The line through $(4,7)$ perpendicular to $y = (2/3)x + 1$.
6. Where does the line through $(2,5)$ and $(5,-4)$ intersect the x-axis?

Answers to Exercises III

A: 1.(a) 8 (b) $3x+11$ (c) $5/2$ (d) $x^2 + (1/x^2)$ (e) undefined

(h) $g(3x+2) = 3x + 2 + 1/(3x+2)$ (i) $-3 + 2 + 1/(-3+2) = -2$

(j) $g(-1) = -2$, so $f(g(-1)) = f(-2) = -4$.

2. $(2/3)(3+1)^{3/2} - (2/3)(1)^{3/2} = 14/3$

3.(a) The set of real numbers greater than or equal to 2 $= \{x \geq 2\}$

(b) The set of real numbers greater than 2 $= \{x > 2\}$

(c) The set where $f(x)$ is not zero $= \{ x \neq -2/3 \}$

(d) The set where g is defined $= \{ x \neq 0 \}$

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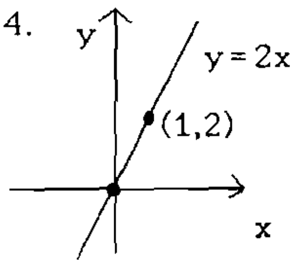

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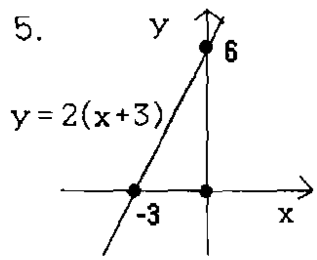

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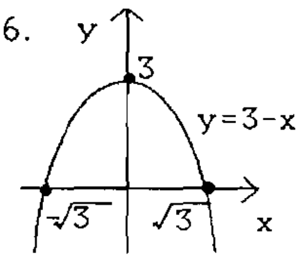

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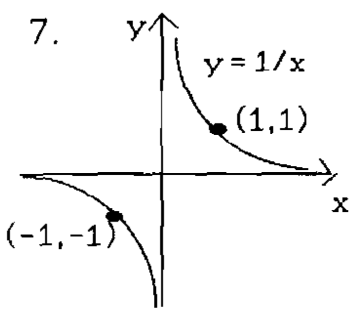

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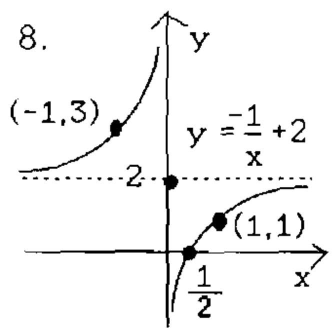

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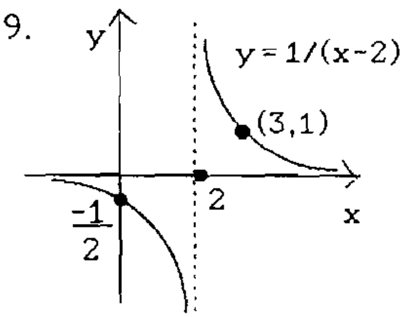
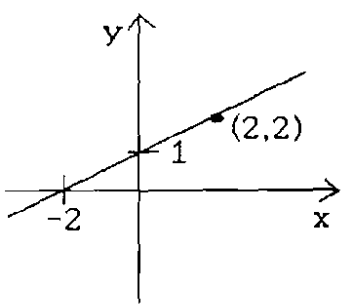
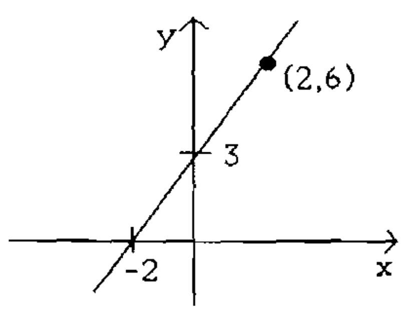
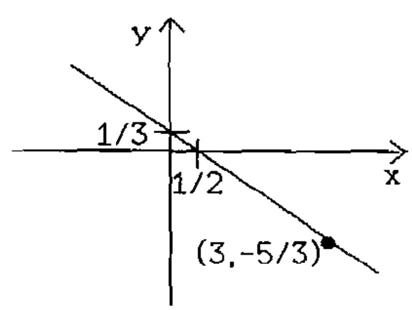


B: 1.(a) $y = \frac{1}{2}x + 1$ (b) $y = \frac{3}{2}x + 3$ (c) $y = -\frac{2}{3}x + \frac{1}{3}$







2. $y - 1 = 2(x - 4)$

3. slope $= m = \dfrac{1-2}{3+1} = \dfrac{-1}{4}$; eq'n: $y - 2 = \dfrac{-1}{4}(x+1)$ or $y - 1 = \dfrac{-1}{4}(x-3)$

4. $y - 7 = 3(x-4)$

5. $y - 7 = -(3/2)(x-4)$

6. Slope $= -3$ so line is $y-5 = -3(x-2)$; intersects x-axis when $y = 0$: $-5 = -3(x-2)$, so $x = 11/3$.